DEVELOPMENTS OF A PFAFFIAN.

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1. The *fundamental* development of a Pfaffian is, of course, that which Jacobi used for the purposes of a *definition*, viz., in umbral notation,

$$\begin{aligned} |123456| \;=\; & 12\cdot|3456| - 13\cdot|2456| + 14\cdot|2356| \\ & - 15\cdot|2346| + 16\cdot|2345|, \end{aligned}$$

or, more concretely,

$$\left|\begin{array}{ccccc} a_2 & a_3 & a_4 & a_5 & a_6 \\ & b_3 & b_4 & b_5 & b_6 \\ & & c_4 & c_5 & c_6 \\ & & & d_5 & d_6 \\ & & & & e_6 \end{array}\right| = a_2\cdot\left|\begin{array}{ccc} c_4 & c_5 & c_6 \\ & d_5 & d_6 \\ & & e_6 \end{array}\right| - a_3\cdot\left|\begin{array}{ccc} b_4 & b_5 & b_6 \\ & d_5 & d_6 \\ & & e_6 \end{array}\right| + a_4\cdot\left|\begin{array}{ccc} b_3 & b_5 & b_6 \\ & c_5 & c_6 \\ & & e_6 \end{array}\right|$$

$$- a_5\cdot\left|\begin{array}{ccc} b_3 & b_4 & b_6 \\ & c_4 & c_6 \\ & & d_6 \end{array}\right| + a_6\cdot\left|\begin{array}{ccc} b_3 & b_4 & b_5 \\ & c_4 & c_5 \\ & & d_5 \end{array}\right|,$$

the fifteen terms of the development of the Pfaffian of the third degree being got in five sets of three terms each. The analogous development of a determinant, it may be remembered, viz.,

$$|a_1b_2c_3d_4| = a_1|b_2c_3d_4| - a_2|b_1c_3d_4| + a_3|b_1c_2d_4| - a_4|b_1c_2d_3|$$

has also been used for the purposes of a definition, the first to do so being Vandermonde (1771).

2. In the second place, since

$$\left|\begin{array}{ccccc} a_2 & a_3 & a_4 & a_5 & a_6 \\ & b_3 & b_4 & b_5 & b_6 \\ & & c_4 & c_5 & c_6 \\ & & & d_5 & d_6 \\ & & & & e_6 \end{array}\right| = a_2\left|\begin{array}{ccc} c_4 & c_5 & c_6 \\ & d_5 & d_6 \\ & & e_6 \end{array}\right| + \left|\begin{array}{ccccc} \cdot & a_3 & a_4 & a_5 & a_6 \\ & b_3 & b_4 & b_5 & b_6 \\ & & c_4 & c_5 & c_6 \\ & & & d_5 & d_6 \\ & & & & e_6 \end{array}\right|,$$

and since the Pfaffian on the extreme right can be expressed as an aggregate of products, each of which has for its first factor one of the determinants of the set

$$\left\| \begin{matrix} a_3 & a_4 & a_5 & a_6 \\ b_3 & b_4 & b_5 & b_6 \end{matrix} \right\|,$$

and for its second factor one of the remaining elements of the Pfaffian,* we obtain

$$\left| \begin{array}{ccccc} a_2 & a_3 & a_4 & a_5 & a_6 \\ & b_3 & b_4 & b_5 & b_6 \\ & \cdots & \cdots & \cdots & \cdots \\ & & & & e_6 \end{array} \right. = a_2 \left| \begin{array}{ccc} c_4 & c_5 & c_6 \\ & d_5 & d_6 \\ & & e_6 \end{array} \right. \begin{aligned} - |a_3\, b_4|.e_6 + |a_3\, b_5|.d_6 - |a_3\, b_6|.d_5 \\ - |a_4\, b_5|.c_6 + |a_4\, b_6|.c_5 \\ - |a_5\, b_6|.c_4, \end{aligned}$$

the fifteen terms of the development being now got in one set of three and six sets of two. $(\mathrm{E}_{\mathrm{I}})$

3. In the third place, since neither a_3 nor b_3 can occur in the same term with a_2, we have from the second line of § 2

$$\left| \begin{array}{ccccc} a_2 & a_3 & a_4 & a_5 & a_6 \\ & b_3 & b_4 & b_5 & b_6 \\ & \cdots & \cdots & \cdots & \cdots \\ & & & & e_6 \end{array} \right. = a_2 \left| \begin{array}{ccc} c_4 & c_5 & c_6 \\ & d_5 & d_6 \\ & & e_6 \end{array} \right. - a_3 \left| \begin{array}{ccc} b_4 & b_5 & b_6 \\ & d_5 & d_6 \\ & & e_6 \end{array} \right. + \left| \begin{array}{ccccc} \cdot & \cdot & a_4 & a_5 & a_6 \\ & \cdot & b_4 & b_5 & b_6 \\ & & c_4 & c_5 & c_6 \\ & & & d_5 & d_6 \\ & & & & e_6 \end{array} \right. ,$$
$$+ b_3 \left| \begin{array}{ccc} a_4 & a_5 & a_6 \\ & d_5 & d_6 \\ & & e_6 \end{array} \right.$$

or, if we use A_2, A_3, B_3 for the complementary minors of a_2, a_3, b_3, and put for the Pfaffian on the extreme right its equivalent the determinant $- \mid a_4\, b_5\, c_6 \mid$, there results

$$\left| \begin{array}{ccccc} a_2 & a_3 & a_4 & a_5 & a_6 \\ & b_3 & b_4 & b_5 & b_6 \\ & \cdots & \cdots & \cdots & \cdots \\ & & & & e_6 \end{array} \right. = \begin{aligned} a_2 A_2 - a_3 A_3 - \mid a_4\, b_5\, c_6 \mid . \\ + b_3 B_3 \qquad\qquad\qquad \end{aligned} \qquad (\lambda_2)$$

This, however, by reason of the low degree of the Pfaffian on the left, is a defective illustration of the form of development now reached. Taking, instead, the Pfaffian of next higher degree we have

$$\left| \begin{array}{cccc} a_2 & a_3 & a_4 \ldots & a_8 \\ & b_3 & b_4 \ldots & b_8 \\ & \cdots & \cdots & \cdots \\ & & & g_8 \end{array} \right. = \begin{aligned} a_2 A_2 - a_3 A_3 + \\ + b_3 B_3 \qquad \end{aligned} \left| \begin{array}{ccccc} \cdot & \cdot & a_4 & a_5 \ldots & a_8 \\ & \cdot & b_4 & b_5 \ldots & b_8 \\ & & c_4 & c_5 \ldots & c_8 \\ & & & \cdots & \cdots \\ & & & & g_8 \end{array} \right. ,$$

$$= \begin{aligned} a_2 A_2 - a_3 A_3 - \Sigma(\mid a_4 b_5 c_6 \mid . g_8), \\ + b_3 B_3 \qquad\qquad\qquad\qquad\qquad \end{aligned}$$

* See *Trans. Roy. Soc. Edinburgh*, xl., pp. 49–58.

the 105 terms of the Pfaffian being got in 3 sets of 15, and 10 sets of 6 each. (E_2)

4. We may pursue the process further and obtain a fourth development of a like character, but a drawback then attaches to the result, the fact being that it is impossible now to say that the Pfaffian

$$\left|\begin{array}{cccccc} \cdot & \cdot & a_4 & a_5 & \cdots & a_8 \\ & \cdot & b_4 & b_5 & \cdots & b_8 \\ & & c_4 & c_5 & \cdots & c_8 \\ & & & d_5 & \cdots & d_8 \\ & & & & \cdots\cdots & \\ & & & & & g_8 \end{array}\right|$$

with its three vacant places is equal to

$$\begin{array}{l} a_4A_4 \\ -\,b_4B_4 \\ +\,c_4C_4 \end{array} + \left|\begin{array}{ccccccc} \cdot & \cdot & \cdot & a_5 & a_6 & a_7 & a_8 \\ & \cdot & \cdot & b_5 & b_6 & b_7 & b_8 \\ & & \cdot & c_5 & c_6 & c_7 & c_8 \\ & & & d_5 & d_6 & d_7 & d_8 \\ & & & & \cdots & \cdots & \\ & & & & & & g_8 \end{array}\right|,$$

the cofactors of a_4, $-b_4$, c_4 in the said Pfaffian being not A_4, B_4, C_4, but

$$\left|\begin{array}{ccccc} \cdot & b_5 & b_6 & b_7 & b_8 \\ & c_5 & c_6 & c_7 & c_8 \\ & & e_6 & e_7 & e_8 \\ & & & f_7 & f_8 \\ & & & & g_8 \end{array}\right|, \quad \left|\begin{array}{ccccc} \cdot & a_5 & a_6 & a_7 & a_8 \\ & c_5 & c_6 & c_7 & c_8 \\ & & e_6 & e_7 & e_8 \\ & & & f_7 & f_8 \\ & & & & g_8 \end{array}\right|, \quad \left|\begin{array}{ccccc} \cdot & a_5 & a_6 & a_7 & a_8 \\ & b_5 & b_6 & b_7 & b_8 \\ & & e_6 & e_7 & e_8 \\ & & & f_7 & f_8 \\ & & & & g_8 \end{array}\right|.$$

If, therefore, we insist on retaining $a_4A_4 - b_4B_4 + c_4C_4$, we shall be repeating 9 terms already included in $a_2A_2 - a_3A_3 + b_3B_3$, and must rectify the error by bringing in the product

$$-\left|\begin{array}{ccc} a_2 & a_3 & a_4 \\ & b_3 & b_4 \\ & & c_4 \end{array}\right| \cdot \left|\begin{array}{ccc} e_6 & e_7 & e_8 \\ & f_7 & f_8 \\ & & g_8 \end{array}\right|,$$

the result being

$$\left|\begin{array}{cccccc} a_2 & a_3 & a_4 & a_5 & \cdots & a_8 \\ & b_3 & b_4 & b_5 & \cdots & b_8 \\ & & c_4 & c_5 & \cdots & c_8 \\ & & & \cdots & \cdots & \\ & & & & & g_8 \end{array}\right| = a_2A_2 \begin{array}{l} -\,a_3A_3 \\ +\,b_3B_3 \\ \end{array} \begin{array}{l} +\,a_4A_4 \\ -\,b_4B_4 \\ +\,c_4C_4 \end{array} - \left|\begin{array}{ccc} a_2 & a_3 & a_4 \\ & b_3 & b_4 \\ & & c_4 \end{array}\right| \cdot \left|\begin{array}{ccc} e_6 & e_7 & e_8 \\ & f_7 & f_8 \\ & & g_8 \end{array}\right| + \left| a_5 \ b_6 \ c_7 \ d_8 \right|, \qquad (\lambda_3)$$

where, instead of the 105 terms of the proper development, we have 123, viz., 6 sets of 15 each, 1 set of 9, cancelling a previous 9, and 1 set of 24.

5. If with an initial Pfaffian of still higher degree we write the result of the preceding paragraph, *e.g.*,

$$\left|\begin{array}{rrrr} a_2 & a_3 & a_4 \dots & a_{10} \\ & b_3 & b_4 \dots & b_{10} \\ & & c_4 \dots & c_{10} \\ & & & \dots\dots \\ & & & i_{10} \end{array}\right| = \begin{array}{rrr} a_2A_2 & -\ a_3A_3 & +\ a_4A_4 \\ & +\ b_3B_3 & -\ b_4B_4 \\ & & +\ c_4C_4 \end{array} - \left|\begin{array}{rrr} a_2 & a_3 & a_4 \\ & b_3 & b_4 \\ & & c_4 \end{array}\right| . \left|\begin{array}{rrrrr} e_6 & e_7 & e_8 & e_9 & e_{10} \\ & f_7 & f_8 & f_9 & f_{10} \\ & & & & \dots\dots \\ & & & & i_{10} \end{array}\right|$$

$$+ \left|\begin{array}{rrrrrrr} . & . & . & a_5 & a_6 & \dots & a_{10} \\ & . & . & b_5 & b_6 & \dots & b_{10} \\ & & . & c_5 & c_6 & \dots & c_{10} \\ & & & d_5 & d_6 & \dots & d_{10} \\ & & & & e_6 & \dots & e_{10} \\ & & & & & & \dots\dots \\ & & & & & & i_{10} \end{array}\right| ,$$

and then again as an equivalent for the Pfaffian on the extreme right, put

$$\begin{array}{l} -a_5A_5 \\ +b_5B_5 \\ -c_5C_5 \\ +d_5D_5 \end{array} + \left|\begin{array}{rrrrrrrr} . & . & . & . & a_6 & a_7 & \dots & a_{10} \\ & . & . & . & b_6 & b_7 & \dots & b_{10} \\ & & . & . & c_6 & c_7 & \dots & c_{10} \\ & & & . & d_6 & d_7 & \dots & d_{10} \\ & & & & & e_6 & \dots & e_{10} \\ & & & & & & & \dots\dots \\ & & & & & & & i_{10} \end{array}\right|$$

compensating for this error by inserting

$$\left|\begin{array}{rrr} a_2 & a_3 & a_5 \\ & b_3 & b_5 \\ & & c_5 \end{array}\right| .\text{compl} - \left|\begin{array}{rrr} a_2 & a_4 & a_5 \\ & b_4 & b_5 \\ & & d_5 \end{array}\right| .\text{compl} + \left|\begin{array}{rrr} a_3 & a_4 & a_5 \\ & c_4 & c_5 \\ & & d_5 \end{array}\right| .\text{compl}$$

$$- \left|\begin{array}{rrr} b_3 & b_4 & b_5 \\ & c_4 & c_5 \\ & & d_5 \end{array}\right| .\text{compl},$$

we shall obtain an identity which is of still less value when viewed as giving a development of the given Pfaffian, viz.,

$$\left|\begin{array}{rrrr} a_2 & a_3 & a_4 \dots & a_{10} \\ & b_3 & b_4 \dots & b_{10} \\ & & c_4 \dots & c_{10} \\ & & & \dots\dots \\ & & & i_{10} \end{array}\right| = \begin{array}{rrrr} a_2A_2 & -\ a_3A_3 & +\ a_4A_4 & -\ a_5A_5 \\ & +\ b_3B_3 & -\ b_4B_4 & +\ b_5B_5 \\ & & +\ c_4C_4 & -\ c_5C_5 \\ & & & +\ d_5D_5 \end{array} - \left|\begin{array}{rrr} a_2 & a_3 & a_4 \\ & b_3 & b_4 \\ & & c_4 \end{array}\right| .\text{compl}$$

$$+ \left|\begin{array}{rrr} a_2 & a_3 & a_5 \\ & b_3 & b_5 \\ & & c_5 \end{array}\right| .\text{compl} - \left|\begin{array}{rrr} a_2 & a_4 & a_5 \\ & b_4 & b_5 \\ & & d_4 \end{array}\right| .\text{compl}$$

$$+ \left|\begin{array}{rrr} a_3 & a_4 & a_5 \\ & c_4 & c_5 \\ & & d_5 \end{array}\right| .\text{compl} - \left|\begin{array}{rrr} b_3 & b_4 & b_5 \\ & c_4 & c_5 \\ & & d_5 \end{array}\right| .\text{compl}$$

$$+ \left| a_6 \; b_7 \; c_8 \; d_9 \; e_{10} \right| , \qquad (\lambda_4)$$

for here, instead of the 945 terms of the proper development, we have 10 sets of 105 terms each, 5 sets of 45 terms each, which merely cancel 225 terms of the preceding sets, and 1 final set of 120.

6. Using, as above, the contraction "compl" to stand for the complementary minor of the determinant or Pfaffian which precedes it,* we may formulate as follows the series of general identities thus reached:—

$$\left|\begin{array}{cccc} a_2 & a_3 & \cdots & a_{2n} \\ & b_3 & \cdots & b_{2n} \\ & & & \cdots\cdots \end{array}\right| = a_2\mathrm{A}_2 - \Sigma\{|\,a_3\ b_4\,|.\mathrm{compl}\}, \qquad (\mathrm{E}_1)$$

$$= \begin{array}{l} a_2\mathrm{A}_2 - a_3\mathrm{A}_3 \\ \qquad\quad + b_3\mathrm{B}_3 \end{array} - \Sigma\{|\,a_4\ b_5\ c_6\,|.\mathrm{compl}\}, \qquad (\mathrm{E}_2)$$

$$= \begin{array}{l} a_2\mathrm{A}_2 - a_3\mathrm{A}_3 + a_4\mathrm{A}_4 \\ \qquad\quad + b_3\mathrm{B}_3 - b_4\mathrm{B}_4 \\ \qquad\qquad\qquad\quad + c_4\mathrm{C}_4 \end{array} - \left|\begin{array}{ccc} a_2 & a_3 & a_4 \\ & b_3 & b_4 \\ & & c_4 \end{array}\right|.\mathrm{compl}$$

$$+ \Sigma\{|\,a_5\ b_6\ c_7\ d_8\,|.\mathrm{compl}\} \qquad (\mathrm{E}_3)$$

$$= \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

it being remembered that only the first two identities are unexceptionably effective, giving, as they do, the final development of the Pfaffian without superfluous terms.

It may be noted in passing that the triangular mode of disposing the terms of the first kind on the right is not without advantage, in that it is a help to the formation of the terms of the second and following kinds. Thus in (E_4) having to commence with

$$\begin{array}{cccccccc} a_2.\mathrm{cof} & + & a_3.\mathrm{cof} & + & a_4.\mathrm{cof} & + & a_5.\mathrm{cof} \\ & + & b_3.\mathrm{cof} & + & b_4.\mathrm{cof} & + & b_5.\mathrm{cof} \\ & & & + & c_4.\mathrm{cof} & + & c_5.\mathrm{cof} \\ & & & & & + & d_5.\mathrm{cof} \end{array}$$

it is only necessary to leave out in succession the first, second, ... frame-lines of this quasi-Pfaffian, and we obtain

$$+ \left|\begin{array}{ccc} b_3 & b_4 & b_5 \\ & c_4 & c_5 \\ & & d_5 \end{array}\right|.\mathrm{cof} + \left|\begin{array}{ccc} a_3 & a_4 & a_5 \\ & c_4 & c_5 \\ & & d_5 \end{array}\right|.\mathrm{cof} + \left|\begin{array}{ccc} a_2 & a_4 & a_5 \\ & b_4 & b_5 \\ & & d_5 \end{array}\right|.\mathrm{cof} + \ldots;$$

in (E_5), the triangular matrix being larger, it is possible to form from

* If strict uniformity of notation were more important than brevity, such a term as "$a_2\mathrm{A}_2$" would have to be replaced by "a_2 . compl." Perhaps the best uniform notation, however, would be got by using a contraction of the word *cofactor*, say the contraction "cof"; the difficulty of indicating what signs are to be +, and what −, would also then be avoided.

it a Pfaffian of the third order, so that the terms of the second kind just given would be followed by

$$\left|\begin{array}{ccccc} a_2 & a_3 & a_4 & a_5 & a_6 \\ & b_3 & b_4 & b_5 & b_6 \\ & & c_4 & c_5 & c_6 \\ & & & d_5 & d_6 \\ & & & & e_6 \end{array}\right| . \text{cof} ;$$

and so on.

It is also worth noting that there is an interesting alternative form which may be substituted for the first three terms in (E_2), (E_3), ..., viz.,

$$\left|\begin{array}{ccc} a_2 & a_3 & a_4 \\ & b_3 & b_4 \\ & & c_4 \end{array}\right| .\text{compl} - \left|\begin{array}{ccc} a_2 & a_3 & a_5 \\ & b_3 & b_5 \\ & & c_5 \end{array}\right| .\text{compl} + \dots - \left|\begin{array}{ccc} a_2 & a_3 & a_{2n} \\ & b_3 & b_{2n} \\ & & c_{2n} \end{array}\right| .\text{compl.}$$

7. When the identity (E_1) is applied to a Pfaffian of the 2nd degree, (E_2) to a Pfaffian of the 3rd degree, and so on, the development in each case ends with only one determinant under the sign of summation: that is to say, we have

$$\left|\begin{array}{ccc} a_2 & a_3 & a_4 \\ & b_3 & b_4 \\ & & c_4 \end{array}\right| = a_2 . c_4 - \left| a_3 \ b_4 \right| \qquad (\lambda_1)$$

and the other special cases marked (λ_2), (λ_3), (λ_4) above, &c. To all except the first two of these there attaches, of course, the blemish attaching to (E_3), (E_4), ... It is greatly magnified, however, if, instead of viewing the λ identities as giving the development of the single Pfaffian on the left, we transform them so as to present an equivalent for the solitary *determinant* on the right.* Not only so, but the blemish then attaches to (λ_1) and (λ_2) also. Thus, taking the case of (λ_2), which then becomes

$$\left|\begin{array}{ccc} a_4 & a_5 & a_6 \\ b_4 & b_5 & b_6 \\ c_4 & c_5 & c_6 \end{array}\right| = \left|\begin{array}{ccccc} a_2 & a_3 & a_4 & a_5 & a_6 \\ & b_3 & b_4 & b_5 & b_6 \\ & & c_4 & c_5 & c_6 \\ & & & d_5 & d_6 \\ & & & & e_6 \end{array}\right| - a_2 \left|\begin{array}{ccc} c_4 & c_5 & c_6 \\ & d_5 & d_6 \\ & & e_6 \end{array}\right| + a_3 \left|\begin{array}{ccc} b_4 & b_5 & b_6 \\ & d_5 & d_6 \\ & & e_6 \end{array}\right| - b_3 \left|\begin{array}{ccc} a_4 & a_5 & a_6 \\ & d_5 & d_6 \\ & & e_6 \end{array}\right| ,$$

* This is the way in which the identities are viewed in a paper by Mr. J. Brill, which has just appeared in the *Proceedings of the London Math. Soc.* (see Vol. I. of Second Series, pp. 103–111). The subject of the paper, it may also be mentioned, might well be overlooked, as the title under which it appears is "On the Minors of a Skew-symmetrical Determinant," whereas the only theorem contained in it is that here (§ 7) illustrated.

we see that to reach the six well-known terms of $| a_4 b_5 c_6 |$ we have to handle 1 set of 15 terms followed by 3 sets of 3 terms each, the last 9 terms appearing merely for the purpose of cancelling 9 of the 15 which precede them; furthermore, six variables occur in the development which have nothing to do with the function sought to be developed.